Variational Methods for Nonstandard Eigenvalue Problems in Waveguide and Resonator Analysis

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Abstract—The nonstandard (general) eigenvalue problem is defined in operator form by $L(\lambda)f=0$ and $B(\lambda)f=0$, where L and B are linear operators, and for a standard problem L is a linear function of the parameter λ and B does not depend on λ. It is shown by examples, that nonstandard problems arise in electromagnetic problems, and a unified variational principle is formulated from which stationary functionals for the nonstandard eigenvalues can be constructed. The examples include: cutoff problem of a waveguide with surface reactance, propagation problem of an azimuthally magnetized ferrite-filled waveguide, the cutoff problem of a corrugated waveguide and the problem of a material insert in a resonator. It is demonstrated with these simple but nontrivial examples that the present method leads to a good engineering accuracy with very elementary test functions.

I. Introduction

THE VARIATIONAL METHOD is a very effective approximative method applicable in electromagnetic problems. Its power has been clearly demonstrated in the classical works by Harrington and Collin [1], [2]. By simple test functions one can approximate complicated field problems and actually, without solving the field problem itself, obtain highly accurate approximations for interesting parameters of the problem. The eigenvalues of a problem are recognized as important physical parameters and their knowledge often is the main subject of the problem.

Examples of eigenvalues in microwave engineering for which variational methods have been applied are the resonance frequency of a resonator and the propagation factor of a waveguide. Methods treating these problems have been growing more powerful over the years in that more general problems can be solved with less effort. In 1956, Berk [3] derived variational principles for waveguide problems in terms of six scalar field components (the ***EH*** formalism), valid for general lossless anisotropic and inhomogeneous media. In 1971, English and Young [4] obtained the same in terms of three components (the ***E*** formalism). However, because the interesting parameter, the propagation factor β, appeared in their functional equation in quadratic form with powers β and β^2, they had to apply the variational method in a reverse way: solve for the frequency, which is normally known, in terms of the porpagation factor, which is normally unknown. The same defect appears in further studies on the subject [5]–[7]. The Hertzian potential approach [2], leading to just two scalar field components, has been applied to only the most elementary problems, because for more complicated problems the resulting eigenvalue equations are not of the standard type, either because the eigenvalue does not appear in a linear form or it is present in the boundary or interface conditions.

In this paper, the eigenvalue problem is defined in a less restrictive manner so that different parameters involved in the problem can be interpreted as eigenvalues. These more general eigenvalues are called nonstandard eigenvalues and a more exact definition is given in Section II. In the same section, a unified theory for obtaining stationary functionals for different nonstandard eigenvalues, based on a mathematical principle, is presented. Previously, Morishita and Kumagai [6] gave a unified principle based on a physical Lagrangian function. The present theory, however, is more general because it embraces both reciprocal and lossless problems, whereas [6] was limited to only lossless problems. Moreover, the present theory allows for nonstandard eigenvalues, which may be any scalar parameters of the problem. The problems are classified in terms of the complexity of their functional equation. Because there may exist many parameters each recognizable as a nonstandard eigenvalue of the problem, there thus may exist different functionals giving a choice of methods of different complexity in solving the same problem.

Several examples are presented in the remaining Sections III–VI. The examples are chosen as simple as possible, yet nontrivial, to elucidate different aspects of the theory.

In Section III, the cutoff frequency problem of a waveguide with reactance boundaries is studied. Different formulations of the problem are first compared: the ***EH*** formalism leads to a standard eigenvalue problem, which is complicated, whereas the Hertzian potential formalism results in a nonstandard problem with simple application, as is seen by an example. Here, the eigenvalue was the cutoff frequency of the guide. If the boundary reactance, instead, is considered as a nonstandard eigenvalue of the problem, the problem is seen to reduce to a linear one, which is still simpler to handle, and an analytic result is obtained in our example instead of a set of curves.

In Section IV, we consider the azimuthally magnetized ferrite-filled waveguide propagation problem, which is in-

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teresting because it is of a nonstandard form in all its parameters and the solutions have been presented in terms of highly complicated special functions. With a simple polynomial test function, a solution with an accuracy of a few percent is obtained.

In Section V, the corrugated waveguide is being studied, for which the cutoff frequency problem is of the nonstandard form and the resulting functional equation for the eigenvalue is transcendental. Conceiving the depth of the corrugations as a nonstandard eigenvalue, an explicit stationary functional, however, can be constructed, and applying a simple test function, an analytic approximating function is obtained for the depth of corrugations for a certain cutoff frequency.

In the final Section VI, we study a cavity with a homogeneous insert. The dielectric susceptibility of the insert is first treated as a nonstandard eigenvalue of the problem, and a functional for it is derived, which should be useful in microwave diagnostics because it directly gives use the value of the interesting parameter. The functional is tested with an example. If the insert also has magnetic susceptibility, a stationary functional is derived for each of the unknown susceptibilities. Finally, any geometrical measure of the insert can be conceived as a nonstandard eigenvalue of the problem. A functional equation for a simple problem is considered to find a measure of the dielectric body inserted in the resonator.

The examples presented in this study are kept at the simplest level possible as to demonstrate the power of the principle, yet the error level is sufficiently low for most engineering practice.

II. Theory

The nonstandard (general) eigenvalue problem can be expressed in the form

$$L(\lambda)f=0 \tag{1}$$

$$B(\lambda)f=0. \tag{2}$$

Here $L(\lambda)$ is a linear operator (typically a differential or integral operator) depending on a parameter λ, which is a complex scalar. The additional operator $B(\lambda)$, is also linear and generally also depends on the same parameter. The second equation (2) may be absent, as in the case when L is an integral operator; for differential operators L the operator B includes the boundary conditions and interface conditions.

The equations possess the solution $f=0$. The question is: are there values of the parameter λ, called eigenvalues, for which there exist other solutions $f\neq 0$? If the operator $L(\lambda)$ has a linear dependence on the parameter: $L(\lambda)=L_0\lambda M_0$, and $B(\lambda)=B_0$ does not depend on λ at all, we call the problem the standard eigenvalue problem. There exist important electromagnetic problems which do not reduce to standard form, examples of which are given in the following sections.

A variational principle can be associated with the problem (1), (2), provided there exists an inner product pair $(\cdot,\cdot)$, $(\cdot,\cdot)_b$, with respect to which the operators L, B are self-adjoint. In fact, it will be supposed that there exists a third linear operator C not depending on the parameter λ, such that the following Green's formula is valid for any functions f, g defined in the domain of the operators L, B

$$(g,Lf)+(Cg,Bf)_b=(Lg,f)+(Bg,Cf)_b. \tag{3}$$

In all cases considered here, the operator C turns out to be the factor 1. The operator C could also be concealed in a more complicated definition of the inner product $(\cdot,\cdot)_b$, which usually coincides with that of $(\cdot,\cdot)$ for a different domain. Necessary conditions for the definition of the two inner products are as follows.

1) Additivity

$$(g,(f_1+f_2))=(g,f_1)+(g,f_2)$$
$$((g_1+g_2),f)=(g_1,f)+(g_2,f).$$

2) Completeness

$$(g,f)=0, \quad \text{for all } f \text{ implies } g=0$$
$$(g,f)=0, \quad \text{for all } g \text{ implies } f=0.$$

3) Symmetry

$$(g,f)=(f,g), \qquad \text{for all } f,g$$

or Hermitian symmetry $(g,f)=(f,g)^*$, for all f, g.

These properties must be satisfied by both inner products and the property *2*) implies vanishing in the respective domain. The property *3*) is necessary because of the Green's formula (3). In fact, we could try to define in terms of a symmetric inner product $(\cdot,\cdot)_s$ the following inner product that is neither symmetric nor Hermitian symmetric:

$$(g,f)=\alpha(g,f)_s+\beta(g,f^*)_s+\gamma(g^*,f)_s+\delta(g^*,f^*)_s. \tag{4}$$

Here, $\alpha,\beta,\gamma,\delta$ are fixed real coefficients. It is not difficult to demonstrate that the properties *1*) and *2*) are satisfied by (4) if they are satisfied by the symmetric inner product $(\cdot,\cdot)_s$. However, if we try to apply (4) to Green's formula (3) even in a simpler form $(g,Lf)=(Lg,f)$, which should be satisfied for all f, g, we shall run into two possibilities: either we must have $\beta=\gamma=0$ or $\alpha=\delta=0$, which correspond to the symmetric and Hermitian symmetric inner products, respectively. The self-adjointness of the operators in a symmetric inner product involves reciprocal electromagnetic problems, whereas in a Hermitian symmetric inner product it involves lossless problems. Usually, the variational principle is expressed in terms of a Hermitian inner product [3]–[8], whence a reciprocal and lossy problem is considered nonself-adjoint and a special variational principle is needed. This is, however, unnecessary if a symmetric inner product is chosen.

The variational principle of eigenvalue problems can be derived from a variational expression $F(\lambda;f)$ defined by

$$F(\lambda;f)=(f,L(\lambda)f)+(Cf,B(\lambda)f)_b. \tag{5}$$

It is seen that if f satisfies (1), (2), we have $F(\lambda;f)=0$. (5) might be called the potential functional of the operators

L, B according to [21]. The stationarity of this functional is studied by forming the first variation. If the values of f and λ are changed in this expression by δf and $\delta\lambda$, respectively, the first variation of F can be written as

$$\delta F(\lambda; f) = 2[(\delta f, Lf) + (C\delta f, Bf)_b] + \delta\lambda[(f, L'f) + (Cf, B'f)_b] \qquad (6)$$

applying the Green's formula (3). The operators L' and B' denote derivatives with respect to the parameter λ of the operators L, B. If f satisfies (1), (2) with the parameter λ, (6) gives us a relation between the variations of F and λ. If F is kept constant $F(\lambda; f) = 0$, we have $\delta F = 0$ and from (6) also $\delta\lambda = 0$ (unless by chance the bracketed term is zero). Hence, if we solve for λ the equation

$$F(\lambda; f) = (f, L(\lambda)f) + (Cf, B(\lambda)f)_b = 0 \qquad (7)$$

the arising functional $\lambda = J(f)$ is stationary when f is a solution of (1), (2) and the stationary value of $J(f)$ is the value of the corresponding parameter, the nonstandard eigenvalue.

We may also study whether the converse is true, i.e., whether every stationary point of the function $J(f)$ thus formed corresponds to a solution of (1), (2). In fact, if $F(\lambda; f) = 0$ and $\delta J(f) = 0$ when $f = f_0$, $\lambda = \lambda_0 = J(f_0)$, we have from (6) $(\delta f, L(\lambda_0)f_0) + (C\delta f, B(\lambda_0)f_0)_b = 0$ for any δf. If C is an operator having an inverse in the domain of the second inner product and if values of δf can be chosen independently in the domains of the two inner products, from the completeness property of the inner products we can conclude that $L(\lambda_0)f_0 = 0$ and $B(\lambda_0)f_0 = 0$, or that (1), (2) are satisfied..

In view of the preceding, it is clear that if an explicit expression $J(f)$ is obtained by solving (7) for the parameter λ, a functional is obtained, which is stationary in first variations of any solution of (1), (2). Even if we cannot solve (7) for λ, we know that the roots λ_i are stationary and we might try to solve the equation approximately for λ. It is important to realize that λ may be any scalar parameter of the problem, for example, in a resonator problem with a dielectric insert it may be the dielectric constant of the insert or any of its measures, which are not normally conceived as eigenvalues of the resonator problem.

The theory may be generalized if we take into account that usually there are many parameters associated with a problem. If there are N parameters $p_1, p_2, \cdots, p_N$, i.e., we have the linear operators $L(p_1, p_2, \cdots, p_N)$, $B(p_1, p_2, \cdots, p_N)$, the variational principle can be found from (7) for any parameter p_i if we consider all other parameters p_j fixed. If (7) can be solved for $\lambda = p_i$, what results is a stationary functional $J(p_j; f)$. This is a very useful property, because we are able to obtain the value of a parameter by measuring other parameters but not the field quantity f. For example, in the resonator problem, if we know the measures of the insert, and measure the resonance frequency, the functional may be applied to obtain the dielectric parameter of the insert, which is of great importance in microwave diagnostics. Since there is no perturbational approach applied here, the measurement setup is loaded with less restrictions.

Alternatively, we might be interested about the measures of a known dielectric sample, in which case a functional for any measure can be derived from the previous principle.

To generalize even more, there might exist more than one parameter whose value we wish to know. Let us assume that there are two such parameters p_1 and p_2. For fixed parameter $p_3, \cdots, p_N$ values there exist one equation for the determination of p_1 and p_2 in (7). Because this is not enough, we have to know another set of other parameters, say $p'_3, \cdots, p'_N$. Thus, there exist two equations from which we might try to solve the unknown parameters p_1 and p_2. If this can be done, there result two functionals, whose stationary values are these parameter values. This can be generalized to more unknown parameters. As an example we may think of an insert in a resonator with unknown ϵ and μ. Measuring two resonance frequencies, two equations (7) for ϵ and μ arise, from which a functional for each can be derived by elimination of the other parameter. The inhomogeneous dielectrical medium of a resonator could be approximated by a piecewise homogeneous medium and the dielectric constants of each piece can be conceived as parameters p_i, which together with the frequency as an additional parameter are nonstandard eigenvalues of the problem. If n different resonance frequencies are measured, there exist n equations for the other n parameters.

Finally, we may consider (7) as a variational principle for the functional $F(\lambda; f)$ if the value of λ is kept fixed. The stationary value of this functional is of course known to be zero. The functional F is, however, unnecessary, because we may use any existing stationary functional found in a book, for a standard eigenvalue, solve it for any parameter in the functional, and obtain a stationary functional for that parameter, which is by definition a nonstandard eigenvalue of the problem. The following might serve as a hierarchical classification of the different types of nonstandard eigenvalue problems in terms of easiness of solution of the functional equation (7).

1) Standard eigenvalue problem, $L(\lambda)$ is a linear function and B does not depend on λ. Equation (7) is a linear equation on λ and an explicit stationary function can be written

$$J(f) = \frac{(f, L_0 f) + (Cf, B_0 f)_b}{(f, M_0 f)}. \qquad (8)$$

2) Nonstandard eigenvalue problem, where both L and B are linear functions of λ. This is as easy to solve as the previous case. Denoting $B(\lambda) = B_0 - \lambda T_0$, we have

$$J(f) = \frac{(f, L_0 f) + (Cf, B_0 f)_b}{(f, M_0 f) + (Cf, T_0 f)_b}. \qquad (9)$$

3) Nonstandard eigenvalue problem, where L and B are at most quadratic functions of λ. In this case, (7) is a quadratic equation and can be solved for λ. Thus, there arise two functionals for the eigenvalue, which are both relevant to the problem. This case is obtained in many

practical problems, as will be seen in some examples in the following sections.

4) Nonstandard problem with an interesting parameter p_1 which cannot be solved from (7) in explicit form because the equation is of higher algebraic or transcendental form. In this case, we might try an approximate solution with the aid of Newton's method or solve for another parameter p_2 involved in the problem if the equation is solvable in explicit form. In the latter case, we can solve p_2 for a set of values p_1, to obtain a relation $p_2(p_1)$ from which the value of the interesting parameter p_1 can be obtained. This method has been applied for a waveguide problem in [4], where the problem is nonstandard in the propagation factor β and standard in the frequency ω^2. Alternatively, we can solve for the functional F for a set of the parameters. Because F should have the stationary value zero, this can be applied to obtain an idea of the convergence of difference approximations.

Finally, we study the applicability of the Rayleigh–Ritz method [2] for nonstandard eigenvalue problems. In linear cases 1) and 2), the method works in the normal way, i.e., it transforms the problem to a standard algebraic eigenvalue problem

$$\underline{\underline{L}}\cdot\underline{f}=\lambda\underline{\underline{M}}\cdot\underline{f}. \qquad (10)$$

In the nonstandard problem of the type 3), however, we have a quadratic algebraic eigenvalue problem

$$(\lambda^2\underline{\underline{A}}+\lambda\underline{\underline{B}}+\underline{\underline{C}})\cdot\underline{f}=0 \qquad (11)$$

which can be called a nonstandard algebraic eigenvalue problem. A problem of this kind arises in circuit theory if we try to find the natural frequencies of a network consisting of frequency independent resistors, inductors, and capacitors. Conversely, we may interpret (11) in terms of an equivalent circuit. The approximate nonstandard eigenvalues in the general case are roots of the algebraic equation

$$\det\left[(\phi_i, L(\lambda)\phi_j)+(C\phi_i, B(\lambda)\phi_j)_b\right]=0 \qquad (12)$$

if the set $\{\phi_i\}$ is used to approximate the unknown field function f. In case 4), this might be of a complicated transcendental form. Any interesting single root can however be found by applying Newton's iteration method.

III. The Waveguide with Reactance Boundaries

As a first simple example we consider a waveguide of any cross section with a surface impedance $Z_s=jX_s$, where X_s is real and independent of the frequency. Considering different formulations of the cutoff problem we see that a nonstandard formulation may lead to a much simpler functional than a standard formulation, Fig. 1.

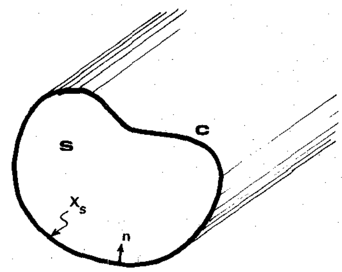

Fig. 1. The waveguide with reactance boundary.

A. $\boldsymbol{E}-\boldsymbol{H}$ Formulation

The cutoff problem of a waveguide is equal to a two-dimensional resonator problem. In fact, the nonpropagating fields do not depend on the z coordinate. The Maxwell's equations with the reactance boundary condition can be written in the abstract form (1), (2) with $L(\omega)=L_0-\omega M_0$, $B(\omega)=B_0$ if we define

$$L_0=\begin{pmatrix}0 & \nabla\times\boldsymbol{I}\\ \nabla\times\boldsymbol{I} & 0\end{pmatrix} \qquad M_0=j\begin{pmatrix}\epsilon & 0\\ 0 & -\mu\end{pmatrix}$$

$$B_0=\begin{pmatrix}0 & 0\\ -\boldsymbol{n}\times\boldsymbol{I} & jX_s\boldsymbol{I}_t\end{pmatrix}. \qquad (13)$$

Here, $\boldsymbol{I}$ is the unit dyadic $\boldsymbol{I}=\boldsymbol{u}_x\boldsymbol{u}_x+\boldsymbol{u}_y\boldsymbol{u}_y+\boldsymbol{u}_z\boldsymbol{u}_z$ and $\boldsymbol{I}_t$ is the transversal unit dyadic $\boldsymbol{I}_t=\boldsymbol{I}-\boldsymbol{nn}$. The inner products are defined by

$$(f_1,f_2)=\int_S(\boldsymbol{E}_1\cdot\boldsymbol{E}_2+\boldsymbol{H}_1\cdot\boldsymbol{H}_2)\,dS$$

$$(f_1,f_2)_b=\oint_C(\boldsymbol{E}_1\cdot\boldsymbol{E}_2+\boldsymbol{H}_1\cdot\boldsymbol{H}_2)\,dC \qquad (14)$$

and the elements f are the vector function pairs $(\boldsymbol{E}(x,y)\boldsymbol{H}(x,y))$ defined on the surface S and its boundary curve C. The operator C in the Green's formula (3) is equal to 1. This is a standard eigenvalue problem for the eigenvalue ω. Hence, we may apply the well-known functional (8), which in this case leads to the following stationary functional:

$$J(\boldsymbol{f},\boldsymbol{g})=\frac{\int(\boldsymbol{f}\cdot\nabla\times\boldsymbol{g}+\boldsymbol{g}\cdot\nabla\times\boldsymbol{f})\,dS-\oint(\boldsymbol{n}\cdot\boldsymbol{f}\times\boldsymbol{g}-jX_s\boldsymbol{g}\cdot\boldsymbol{I}_t\cdot\boldsymbol{g})\,dC}{j\int(\epsilon\boldsymbol{f}\cdot\boldsymbol{f}-\mu\boldsymbol{g}\cdot\boldsymbol{g})\,dS}. \qquad (15)$$

Applying Hermitian symmetric inner product instead of (14) results in a slightly different form. Equation (15) is stationary also for complex X_s, whereas the Hermitian symmetric form is not.

B. $\boldsymbol{E}$ Formulation

The trouble with (15) is that it is too complicated: there are two vector functions $\boldsymbol{f}(x,y)$, $\boldsymbol{g}(x,y)$ to be approximated. A simplification can be obtained if we start from the Helmholtz equation for the electric field. However, in this case the problem is of nonstandard form

$$\nabla\times(\nabla\times\boldsymbol{E})-\omega^2\mu\epsilon\boldsymbol{E}=0, \quad \text{on } S \qquad (16)$$

$$\boldsymbol{n}\times(\nabla\times\boldsymbol{E})-\omega(\mu/X_s)\boldsymbol{E}_t=0, \quad \text{on } C. \qquad (17)$$

Defining the operators by

$$L(\lambda)=\nabla\times\nabla\times\boldsymbol{I}-\lambda^2\mu\epsilon\boldsymbol{I}$$

$$B(\lambda)=\boldsymbol{n}\times(\nabla\times\boldsymbol{I})-\lambda(\mu/X_s)\boldsymbol{I}_t \qquad (18)$$

and the inner products by

$$(f_1, f_2) = \int_S E_1 \cdot E_2 \, dS$$

$$(f_1, f_2)_b = \oint_C E_1 \cdot E_2 \, dC \tag{19}$$

we have $C=1$ in the Green's formula (3) and (7) is of the second degree

$$F(\lambda; f) = -\lambda^2 \int \mu\epsilon f^2 \, dS - \lambda(\mu / X_s) \oint f_t^2 \, dC + \int (\nabla \times f)^2 \, dS = 0. \tag{20}$$

This can be solved for λ in explicit form and the result is a pair of functionals giving as stationary value the cutoff frequency ω

$$J(f) = \frac{1}{2\epsilon X_s} \frac{\oint f_t^2 \, dC}{\int f^2 \, dS} \pm \sqrt{\left(\frac{1}{2\epsilon X_s} \frac{\oint f_t^2 \, dC}{\int f^2 \, dS} \right)^2 + \frac{\int (\nabla \times f)^2 \, dS}{\mu\epsilon \int f^2 \, dS}}. \tag{21}$$

Despite the more complex appearance, the functionals (21) are more attractive than (15) because there is only one vector function f to be approximated. Solving a quadratic equation for a stationary functional was done in Morse and Feshbach [22], but the result was considered more a fortunate accident than a seed for a general method.

C. Hertzian Potential Formulation

The problem can be still simplified if Hertzian potentials are applied. In fact, the electromagnetic field can be expressed in terms of two scalar two-dimensional potential functions for any waveguide mode [2]

$$E(r) = \left[u_z k_c^2 \pi(\rho) - j\beta \nabla \pi(\rho) + jk u_z \times \nabla m(\rho) \right] e^{-j\beta z} \tag{22}$$

$$\eta H(r) = \left[u_z k_c^2 m(\rho) - j\beta \nabla m(\rho) - jk u_z \times \nabla \pi(\rho) \right] e^{-j\beta z}. \tag{23}$$

At cutoff we have $\beta = 0$ and $k_c = k = \omega\sqrt{\mu\epsilon}$ and the problem (16), (17) takes on the form

$$(\nabla^2 + k^2)\binom{\pi}{m} = 0, \qquad \text{on } S \tag{24}$$

$$n \cdot \nabla \pi + k \frac{1}{p} \pi = 0$$

$$n \cdot \nabla m - kpm = 0$$

$$p = X_s / \eta, \qquad \text{on } C. \tag{25}$$

Although there exist no pure TE^z or TM^z modes propagating in a waveguide with general reactance boundaries, from (24), (25) we see that the π and m potentials are independent, whence at cutoff the fields are seen to reduce to TE^z and TM^z modes. This fact can be used to classify the modes into two sets.

Let us concentrate on the TE cutoff problem. From (24), (25) we see that it is a nonstandard eigenvalue problem of the quadratic type. Identifying the operators $L = \nabla^2 + k^2$ and $B = n \cdot \nabla - kp$, the Green's formula (3) can be seen to exist with the operator $C = 1$ and the inner products defined by $(f_1, f_2) = \iint f_1 f_2 \, dS$, $(f_1, f_2)_b = \oint f_1 f_2 \, dC$. Hence, (7) can be solved for the eigenvalue λ

$$J(f) = p \frac{\oint f^2 \, dC}{2 \int f^2 \, dS} \pm \sqrt{\left(p \frac{\oint f^2 \, dC}{2 \int f^2 \, dS} \right)^2 + \frac{\int (\nabla f)^2 \, dS}{\int f^2 \, dS}}. \tag{26}$$

The form of this functional is evidently superior to (21) and (15) in simplicity. Like (21), (26) contains two functionals, which are both relevant to the problem. However, all the information can be obtained from either of them. In fact, looking at the original problem (24), (25), and values of the boundary parameter p, we see in comparing the two functions $J_\pm(p, f)$ that we have the simple relation

$$J_\pm(p, f) = -J_\mp(-p, f) \tag{27}$$

or the values obtained from the functional $J_-(f)$ for the parameter p are obtained from the other functional $-J_+(f)$ for the parameter $-p$. Further, the functional for the TM cutoff can also be reduced to this same functional because we may write

$$J_\pm^{\text{TM}}(p, f) = J_\pm^{\text{TE}}\left(-\frac{1}{p}, f\right). \tag{28}$$

Hence, it suffices to consider only one functional, say $J_+(f)$.

D. The Circular Cylindrical Waveguide

As a simple numerical example we consider a circular cylindrical waveguide with surface reactance $X_s = p\eta$ and radius a, Fig. 2. The Hertzian potential problem can be solved in terms of Bessel functions; in fact the most general solution of either potential in (24), (25) is a linear combination of functions $J_n(k\rho)e^{\pm jn\phi}$. Imposing the boundary condition for the TE mode leaves us with the characteristic equation

$$J_n'(ka) = pJ_n(ka). \tag{29}$$

This equation can be solved for low values of n with high accuracy applying tabulated values of the Bessel function. The eigenvalue ka as a function of the parameter p is given in Fig. 3 for the lowest TE$_{0s}$ modes.

Now we apply the functional $J_+(f)$ in (26). Taking a simple linear approximation of the Hertzian potential function $m(\rho)$

$$f(\rho) = \rho + \alpha a \tag{30}$$

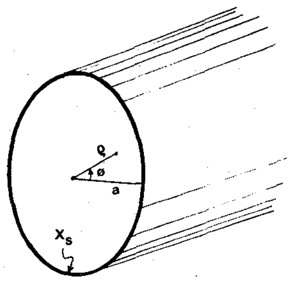

Fig. 2. The reactive circular waveguide.

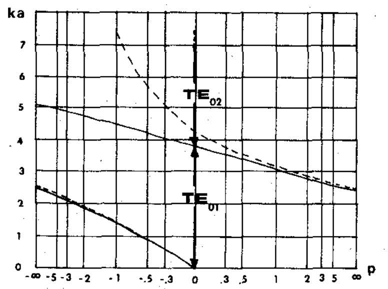

Fig. 3. Relation between cutoff values ka and normalized surface reactance $p = X_s/\eta$ of a circular cylindrical waveguide. Solid line: exact, dashed line: approximate.

we have from (26)

$$aJ_+(\alpha) = -\frac{p(1+\alpha)^2}{\alpha^2+4\alpha/3+1/2} + \sqrt{\frac{p^2(1+\alpha)^4}{(\alpha^2+4\alpha/3+1/2)} + \frac{1}{\alpha^2+4\alpha/3+1/2}}. \tag{31}$$

The extremal values of this function with respect to α can be found very easily with the aid of a programmable calculator. For example, for $p=1$ we have a maximum for J_+ at $\alpha = -0.795$ corresponding to the cutoff value $ka = 3.188$, which is in error by 2.4 percent. The approximate values obtained from (31) are depicted in Fig. 3 by a dashed line. It is seen that the approximation by a linear function is fair for the TE_{01} mode but fails for higher modes, as expected.

Here we have considered only positive values of the functional. The negative values are obtained if the diagram is rotated 180° around the origin. The eigenvalues for the TM modes are obtained if both halves of the diagram are interchanged. If the figure is drawn on a cylinder in such a way that the $p = -\infty$ line coincides with the $p = +\infty$ line, this last operation equals 180° rotation of the figure on the cylinder.

E. A Functional for the Reactance

There are two parameters involved in this simple example, namely the cutoff frequency and the boundary reactance parameter p. We could consider this problem with fixed cutoff frequency and find the corresponding values of p, which are nonstandard eigenvalues of the problem. The

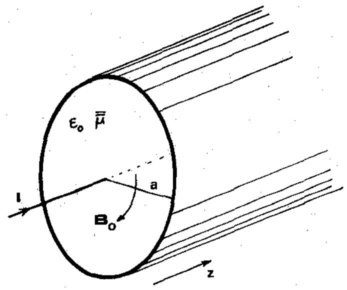

Fig. 4. The ferrite-filled circular waveguide.

problem is of the type 2) for the parameter p, because $B(p)$ is a linear function and L is a constant. Hence, a simpler functional (9) can be applied with $M_0 = 0$, $L_0 = L$, $B_0 = \boldsymbol{n}\cdot\nabla$ and $T_0 = k$. The functional reads

$$J(f) = \frac{\int (\nabla f)^2\, dS - k^2 \int f^2\, dS}{k \oint f^2\, dC}. \tag{32}$$

Applying the same approximation (30) we have

$$J(\alpha) = \frac{1-(ka)^2(\alpha^2+4\alpha/3+1/2)}{2(ka)(1+\alpha)^2} \tag{33}$$

for which we can find the stationary value of α in an analytical form $\alpha = -(3/(ka)^2 + 1/2)$. Thus, an analytic approximation for the parameter p in terms of ka exists

$$p \simeq \frac{-ka\left((ka)^2-18\right)}{6\left((ka)^2-6\right)}. \tag{34}$$

If we wish to have an approximation for the function $ka(p)$, we can approach by treating ka fixed and p a nonstandard eigenvalue, whereas to know the cutoff frequency for a certain value of p, it is simpler to apply the more complicated functional (26), once than, (32), for many times.

IV. Waveguide with Azimuthally Magnetized Ferrite

A circular waveguide filled with ferrite medium magnetized to remanence with the aid of an axial current pulse has proved useful in microwave phase shifting devices [9]–[12]. The operating mode is TE_{01}, which is rotationally symmetric and the propagation factor β depends on the direction (handedness) of the magnetization with respect to the direction of propagation. The magnetization in the ferrite can be reversed with a current pulse, whence the propagation factor is changed in a very short period of time, Fig. 4.

The permeability dyadic of the ferrite in remanence can be written in the form [9], [11]

$$\boldsymbol{\mu} = \mu_0(\boldsymbol{I} + jp\boldsymbol{u}_\phi \times \boldsymbol{I}), \qquad p = \gamma M_\phi/\omega. \tag{35}$$

Here, M_ϕ is the magnetization in the ϕ direction (negative if in the $-\phi$ direction) and γ is the gyromagnetic ratio of the medium. The magnetization is assumed homogeneous, but μ is not constant because it depends on the angle ϕ. The mathematical problem of TE_{10} mode propagation in the guide can be expressed in terms of a single scalar component, for instance E_ϕ as in [9] or H_z [11]. The latter obeys the following differential equation and boundary conditions:

$$\nabla^2 H_z(\rho)+\left(k^2-k^2p^2-\beta^2-\frac{p\beta}{\rho}\right)H_z(\rho)=0,$$
$$H_z'(0)=H_z'(a)=0. \quad (36)$$

The conductor filament on the axis is assumed infinitely thin and for the TE_{10} mode it does not perturb the field and can in fact be neglected if the field is assumed finite on the axis.

Equation (36) can be exactly solved in terms of Kummer and Tricomi confluent hypergeometric functions of pure imaginary argument [9], [11], and the zeros of these functions have been tabulated by Ivanov [11], [12].

The problem is, however, more directly attacked by variational methods. From (36) we see that there are three parameters involved: k, p, and β. It is also seen that the operator is of the nonstandard form in all of these parameters: it is of second degree in p and β, whereas if the ω dependence of p is taken into account, (7) would be of the third degree in k or ω. Normally there are two kinds of questions posed: what is the propagation factor β for certain values of other parameters and for what value of magnetization M_ϕ or parameter p do we have a certain propagation factor? The last question can be posed in a more specific form: for what value of opposite magnetization $\pm M$ do we have a certain average propagation factor $(\beta_+ + \beta_-)/2$ and a certain difference of propagation factors $\beta_+ - \beta_-$?

The functional is easily obtained for β from (7) and the result is

$$J(f)=-\frac{p\int f^2\,d\rho}{2\int f^2\rho\,d\rho}\pm\sqrt{\left(\frac{p\int f^2\,d\rho}{2\int f^2\rho\,d\rho}\right)^2+k^2-p^2k^2-\frac{\int (f')^2\rho\,d\rho}{\int f^2\rho\,d\rho}}. \quad (37)$$

As a numerical example we may take the lowest degree power function satisfying the correct boundary conditions (36). From $f'=3\rho(\rho-a)$ we have

$$f(\rho)=\rho^3-\frac{3}{2}a\rho^2+\alpha \quad (38)$$

where α is a parameter. Varying α we can find the stationary value of $J(f)$ with a programmable calculator in just a few steps. In Fig. 5 the values of (37) with + sign are shown in dashed line and comparison with the exact values [12] can be made. The curves are calculated for $ka=16$ for

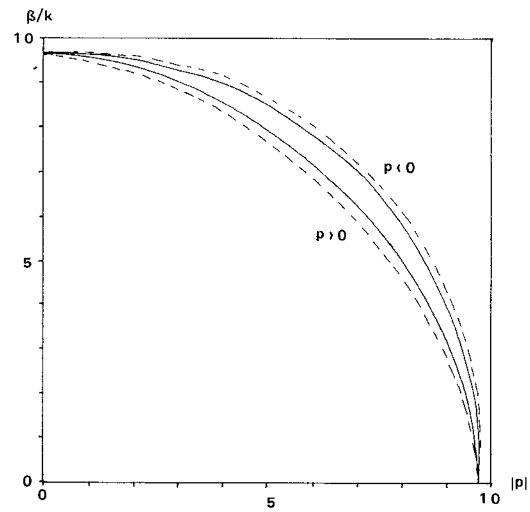

Fig. 5. Normalized propagation constant β/k in a ferrite-filled waveguide for opposite values of the magnetization parameter p. Solid line: exact [12], dashed line: approximate. $ka=16$.

different values of the magnetization parameter p. It is found that for $|p|<0.5$, the error is less than about 3 percent, which is enough for most engineering purposes. For higher values of $|p|$ the third-degree polynomial apparently is unable to approximate the field distribution accurately enough. Approximation for the field is obtained from (38) by substituting the α value at the stationary point.

The functional for the other nonstandard eigenvalue p can be obtained very easily from (37) applying a transformation. In fact, if in the original problem (36) we replace β by kp and p by β/k, the problem does not change at all. Hence, the same transformation can be made in the functional (37) and the resulting $J(f)$ gives us as a stationary value an approximation to kp. In fact, we can use the resulting diagram (Fig. 5) and scale it according to this transformation, whence we would get the same diagram. The diagram is, hence, symmetric at the 45° line and it suffices to determine only half of it.

It is the merit of this method that more complicated geometries, for example the one involving a ferrite rod on the axis of the circular waveguide with a dielectric sheath, can be handled with just a little more complication in the functional, whereas the exact formulation involves a characteristic equation with ratios of both Bessel's functions and Kummer's hypergeometric functions of imaginary argument, which make the analysis very time consuming [12].

V. The Corrugated Waveguide

Corrugated waveguides and corrugated horns have been applied as feed elements for parabolic reflector antennas because of their rotational symmetric radiation pattern and low crosspolarization [13], [14] (see Fig. 6). The corrugated waveguide has been studied extensively [15]–[18] in both rectangular and circular cylindrical geometry applying exact formulation with special functions. When the corrugations are very thin, i.e., the period d is much smaller than the wavelength in the guide, the anisotropic surface imped-

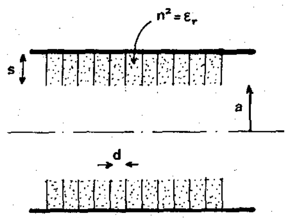

Fig. 6. The corrugated dielectrically loaded circular waveguide.

ance approximation can be applied [19]. The waveguide problem differs from the problem considered in Section III in that the surface impedance is now inherently frequency dependent. The problem is a nonstandard eigenvalue problem leading to a transcendental equation for λ.

Let us consider a circular waveguide with dense corrugations, possibly filled with dielectric material. If the dielectric constant ϵ_r is high enough, the depth of the corrugations can be made small: $s \ll a$, whence we can approximate the true boundary impedance condition involving Bessel's functions [18] by those valid for plane surfaces [19]

$$E_z/H_\phi = -j\frac{\eta}{n}\tan(kns) \qquad E_\phi/H_z = 0, \qquad \text{at } \rho = a. \tag{39}$$

Here, n denotes the refraction index $=\sqrt{\epsilon_r}$. For simplicity, we only try to find out the cutoff frequencies for the lowest modes. Expressing the fields in terms of the Hertzian potentials (22), (23) we have for $\beta = 0$ and $k_c = k$ the problem [20]

$$(\nabla^2 + k^2)(\pi \quad m) = 0, \qquad \text{on } S \tag{40}$$

$$kn\cdot\nabla\begin{pmatrix}\pi\\ m\end{pmatrix} + jk^2\begin{pmatrix}\eta/Z_{tt} & 0\\ 0 & Z_{zz}/\eta\end{pmatrix}\begin{pmatrix}\pi\\ m\end{pmatrix} = 0, \quad \text{on } C \tag{41}$$

$$Z_{tt} = j\frac{\eta}{n}\tan(kns) \qquad Z_{zz} = 0. \tag{42}$$

The boundary conditions (41) do not couple the potential functions, whence at cutoff there exist pure TE and TM modes. This could be applied for the classification of the modes. The TE mode cutoff problem is simple, because it is the same as that for the smooth conducting guide. For the TM mode we have the boundary condition

$$\boldsymbol{n}\cdot\nabla\pi + kn\cot(kns)\pi = 0. \tag{43}$$

Thus, the parameters k, n, and s are in a transcendental function and, hence, the equation (7) is transcendental

$$F(k,n,s;f) = k^2\int_S f^2\,dS - \int_S(\nabla f)^2\,dS - kn\cot(kns)\oint_C f^2\,dC = 0. \tag{44}$$

An explicit functional cannot be found for k nor n but, instead, for s we can write

$$J(f) = \frac{1}{kn}\cot^{-1}\left(\frac{-\int(\nabla f)^2\,dS + k^2\int f^2\,dS}{kn\oint f^2\,dC}\right). \tag{45}$$

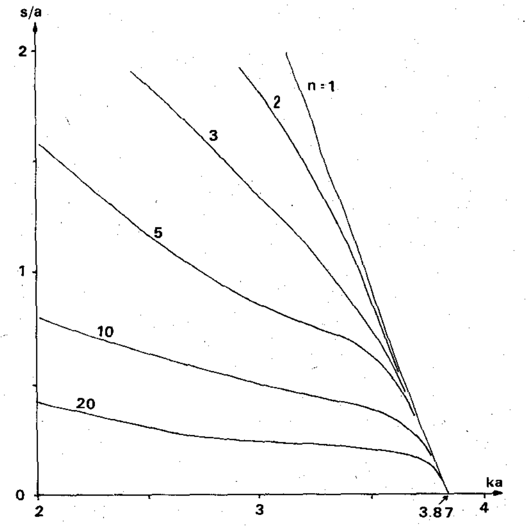

Fig. 7. Relation between the cutoff value ka, normalized depth of corrugation s/a and index of refraction $n=\sqrt{\epsilon_r}$ in the circular corrugated waveguide of Fig. 6.

From this functional it is possible to obtain the value s of the corrugation depth, giving us a given cutoff wavenumber k for a certain mode if the potential corresponding to that mode is approximated in (45). The lowest TM cutoff mode is the mode designated HE_{11} and at cutoff its field resembles that of TM_{11} of the smooth waveguide. For a trial function we take one with $\cos\phi$ dependence on the azimuthal coordinate and the simplest polynomial of ρ with a parameter and vanishing on the axis, i.e., $\rho(\rho-\alpha)$

$$f(\rho,\phi) = \rho(\rho-\alpha)\cos\phi. \tag{46}$$

If this is substituted in (45) and integrations carried out, we look for the stationary point $(d/d\alpha)J(\alpha) = 0$, whence we have from the resulting equation $k\alpha = (2/3)ka + (5/ka)$, which substituted in the $J(\alpha)$ expression gives us

$$s/a \simeq \frac{1}{nka}\cot^{-1}\left(\frac{1}{nka}\left(\frac{(ka)^2(2(ka)^2-75)}{20((ka)^2-15)} - 1\right)\right). \tag{47}$$

The validity of the approximate expression (47) is obviously limited for low values of s/a, because of the approximation (39). In Fig. 7 we see values of s/a for different values of ka and $n=\sqrt{\epsilon_r}$ calculated from (47). It is seen that all curves go through the point $s/a = 0$, $ka = 3.87$, which corresponds to the cutoff value of the smooth waveguide for the TM_{11} mode. The true value is 3.832, whence the error is 1 percent. Also, one of the points can be checked from a diagram in [18]. For $a/(a+s) = 0.9$ or $s/a = 0.1111$ we have from (47) the value $ka = 3.438$, which lies on the curve in Fig. 3 of [18] within reading accuracy.

The possibility of obtaining an explicit expression (47) for a transcendental nonstandard eigenvalue problem is, of course, accidental and if we had not made the starting approximation (39), we could not have arrived at such a simple result. In that case, we could have treated all the parameters k, n, s known, and looked for the stationary points of the functional $F(k,n,s;f)$ itself from (44).

VI. The Inhomogeneous Resonator

As a final example we consider a cavity resonator with inserted unknown medium and mainly concentrate on microwave diagnostic problems, i.e., determination of material properties by measurements of the resonance frequencies and Q values of the resonator (see Fig. 8). The material parameters can be conceived as nonstandard eigenvalues of the problem and stationary functionals can be constructed directly for the unknown parameters instead of measured parameters, which reduces the amount of calculations needed.

A. Dielectric Insert

In this first example we consider a resonator with a dielectric insert and derive a stationary functional for the dielectric constant. To test the principle stated in Section II, viz., that any correct functional stationary in one parameter (eigenvalue) can be applied to derive a stationary function for another parameter, we borrow a functional for the quantity ω^2 from [1, eqs. (7)–(45)]

$$\omega^2 = \frac{\int \mu^{-1}(\nabla \times \boldsymbol{E})^2 \, dV + 2\oint \boldsymbol{n}\cdot\left((\mu^{-1}\nabla\times\boldsymbol{E})\times\boldsymbol{E}\right) dS}{\int \epsilon \boldsymbol{E}^2 \, dV}. \tag{48}$$

Here, the volume integrals extend over the whole resonator and the surface integral over the resonator surface. The test function $\boldsymbol{E}$ is assumed continuous in V, otherwise an additional surface term would appear. Both ϵ and μ may be functions of $\boldsymbol{r}$. Assuming $\mu=\mu_0$ constant and $\epsilon=\epsilon_0(1+\chi(\boldsymbol{r}))$ such that the susceptibility χ is constant in a volume V_1 and $=0$ outside V_1, we may solve (48) for χ

$$\chi = \frac{\int (\nabla\times\boldsymbol{E})^2 \, dV - k^2\int \boldsymbol{E}^2 \, dV + 2\oint \boldsymbol{n}\cdot\boldsymbol{E}\times(\nabla\times\boldsymbol{E})\, dS}{k^2\int_{V_1} \boldsymbol{E}^2 \, dV} \tag{49}$$

where the direction of $\boldsymbol{n}$ is inwards. The volume integrals in the numerator are over the whole volume V and $k^2=\omega^2\epsilon_0\mu_0$.

To test this expression we consider a simple example: a rectangular resonator loaded with a dielectric. The lowest mode (TE_{101}) can be written in terms of sine functions and the characteristic equation for k is of the form

$$\frac{\tan\sqrt{(ka)^2-(\pi a/c)^2}\,(1-d/a)}{\sqrt{(ka)^2-(\pi a/c)^2}} = -\frac{\tan\sqrt{\epsilon_r(ka)^2-(\pi a/c)^2}\,(d/a)}{\sqrt{\epsilon_r(ka)^2-(\pi a/c)^2}}. \tag{50}$$

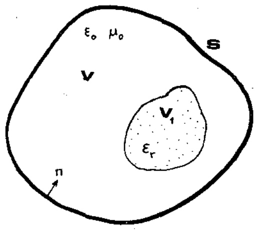

Fig. 8. A resonator with a dielectric insert, $\epsilon_r = 1+\chi$.

Taking the test function $\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{u}_y\sin(\pi z/c)f(x)$, the function (49) is reduced in the following form:

$$J(f) = \frac{\int (f')^2\,dx - \left(k^2-(\pi/c)^2\right)\int f^2\,dx + 2[f(0)f'(0)-f(a)f'(a)]}{k^2\int_0^d f^2\,dx} \tag{51}$$

where other integrals than that in the numerator extend from 0 to a.

As a test we try the simplest power polynomial satisfying the correct boundary conditions $f(0)=f(a)=0$ and containing one parameter

$$f(x)=x(a-x)(1+\alpha x) \tag{52}$$

which substituted in (51) gives us a function of α too complicated to be handled analytically. The optimum, however, is easily obtained with a calculator. For example, for $\epsilon_r=2$ and $c=d=a/2$ we have from (50) the exact value $ka=5.6530$ for resonance. For these figures, from (51) we obtain the stationary point at $\alpha a=0.498$: $\chi=0.9438$, or $\epsilon_r=1.9438$, whence the error is -2.8 percent. At $\epsilon_r=1.1$ the error is only 1.1 percent and it grows for growing ϵ_r.

The functional (49) is more applicable for microwave diagnostics than (48), because the former gives us for a measured value of the resonance frequency ω the interesting susceptance χ value directly, whereas to apply (48), we have to perform a search for the stationary point for many guessed values of χ to obtain the measured ω.

The functional (49) obviously also works for lossy dielectrics, in which case χ becomes complex. For that we have to know the complex resonance frequency. For small losses the real resonance frequency ω_r and the Q value of the resonator can be combined to a complex resonance frequency $\omega=\omega_r(1+j/2Q)$.

Also, application of (49) can be compared with the common perturbational methods. It is evident, that the present method does not impose as many limitations as does the perturbation method to the measurement setup. On the other hand, if the insert is small, the present method is equivalent with the perturbational method [1].

B. Dielectric—Magnetic Insert

The insert of the previous example may also show magnetic susceptibility, in which case we wish to find out two parameters: χ_e and χ_m. Measuring two resonances with the respective frequencies ω_1 and ω_2, we are able to construct stationary functionals for each of the interesting parameters χ_e, χ_m. Starting from the $\boldsymbol{E}-\boldsymbol{H}$ formulation with the operators (13) ($X_s = 0$ in this case), (7) can be evaluated to give

$$\chi_e \epsilon_0 \int_{V_1} \boldsymbol{E}^2\, dV - \chi_m \mu_0 \int_{V_1} \boldsymbol{H}^2\, dV = -\epsilon_0 \int_V \boldsymbol{E}^2\, dV + \mu_0 \int_V \boldsymbol{H}^2\, dV + \frac{1}{j\omega}\int_V (\boldsymbol{E}\cdot\nabla\times\boldsymbol{H} + \boldsymbol{H}\cdot\nabla\times\boldsymbol{E})\, dV + \frac{1}{j\omega}\oint_S \boldsymbol{n}\cdot\boldsymbol{E}\times\boldsymbol{H}\, dS. \tag{53}$$

Denoting the factor functionals as follows:

$$W_e^1(\boldsymbol{E}) = \tfrac{1}{4}\epsilon_0 \int_{V_1} \boldsymbol{E}^2\, dV \qquad W_m^1(\boldsymbol{H}) = \tfrac{1}{4}\mu_0 \int_{V_1} \boldsymbol{H}^2\, dV \tag{54}$$

and the right-hand side of (53) by $4G(\boldsymbol{E}, \boldsymbol{H}; \omega)$, we can write a system of functional equations for χ_e, χ_m for two resonance frequencies ω_1, ω_2 and the corresponding functions approximating the two modes

$$\begin{pmatrix} W_e^1(f_1) & -W_m^1(g_1) \\ W_e^1(f_2) & -W_m^1(g_2) \end{pmatrix} \begin{pmatrix} \chi_e \\ \chi_m \end{pmatrix} = \begin{pmatrix} G(f_1, g_1; \omega_1) \\ G(f_2, g_2; \omega_2) \end{pmatrix}. \tag{55}$$

This system can be solved for the susceptibilities

$$\chi_e = \frac{W_m^1(g_2)G(f_1, g_1; \omega_1) - W_m^1(g_1)G(f_2, g_2; \omega_2)}{W_m^1(g_2)W_e^1(f_1) - W_m^1(g_1)W_e^1(f_2)} \tag{56}$$

$$\chi_m = \frac{W_e^1(f_2)G(f_1, g_1; \omega_1) - W_e^1(f_1)G(f_2, g_2; \omega_2)}{-W_e^1(f_2)W_m^1(g_1) + W_e^1(f_1)W_m^1(g_2)} \tag{57}$$

which are stationary for the correct resonance fields $f_i = \boldsymbol{E}_i$, $g_i = \boldsymbol{H}_i$.

The same method can be applied if the insert is composed of piecewise homogeneous parts, whose dielectric constants can be solved from the functional equation system.

C. The Geometrical Parameters

A functional equation can also be written for any geometrical parameters of the problem. Any of the measures of the resonator itself or the dielectric insert is a possible nonstandard eigenvalue. The problem is to find an explicit form for the parameter in question. Let us consider a simple example for which this can be made, namely, the determination of the thickness d of a dielectric insert in a rectangular cavity as shown in Fig. 9.

The functional (51) can be applied in this case by giving it the value χ and considering the parameter d as the unknown. The trouble is, the unknown in the functional equation is the limit of an integral

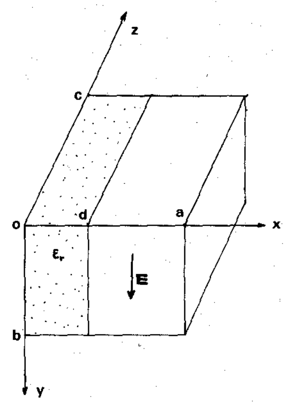

Fig. 9. The dielectrically loaded rectangular resonator.

$$\chi k^2 \int_0^d f^2\, dx = \int_0^a (f')^2\, dx - \left(k^2 - (\pi/c)^2\right) \cdot \int_0^a f^2\, dx + 2[f(0)f'(0) - f(a)f'(a)]. \tag{58}$$

This equation, in fact, defines a functional $d = J(f)$. Of course, we can try to obtain a more explicit expression by transforming the function f to an integrable form. In fact, if instead of the test function f we consider a function $g(x)$ defined by

$$g'(x) = f^2(x) \qquad \text{or } f(x) = \sqrt{g'(x)} \tag{59}$$

(58) can be written in the form

$$g(d) = g(0) + \frac{1}{\chi k^2}\int_0^a \left((g''(x))^2/4g'(x)\right) dx - \frac{k^2 - (\pi/c)^2}{\chi k^2}(g(a) - g(0)) - \frac{1}{\chi k^2}(g''(a) - g''(0)). \tag{60}$$

If the inverse of the function g is known, (60) defines an explicit functional $J(g) = g^{-1}$ (RHS) where RHS denotes the right-hand side of the equation (60).

The application of (60) sets several practical limitations concerning the choice of the test function $g(x)$:

1) $\sqrt{g'(x)}$ should approximate the field function. For a lossless problem, $g(x)$ should be a monotonously increasing positive function;
2) $g(0) = 0$ should be satisfied;
3) to proceed analytically, the function $(g'')^2/g'$ should be analytically integrable;
4) g^{-1} should be performable analytically.

The limitation 4) may not be crucial; if g is simple enough, we can find the stationary value for $g(t)$ and solve the resulting equation for d in some approximate manner.

As an example we consider the test function

$$g(x) = \frac{1}{2}\left(x - \frac{a}{2\pi}\sin\frac{2\pi x}{a}\right) \tag{61}$$

which obviously satisfies 1), 2), and 3), but not 4). Because $g' = (\sin(\pi x/a))^2$, we have $f = \sin(\pi x/a)$, which is a fair approximation. Substituted in (60) we have

$$g(d) = \frac{a}{2\chi k^2}\left((\pi/a)^2 + (\pi/c)^2 - k^2\right). \tag{62}$$

There is no parameter to be optimized in (62), but the value $g(d)$ only involves a quadratic error. Substituting the values of the example in Section VI-A: $\chi = 1$, $c = a/2$, $ka = 5.6530$, we obtain from (62) $g(d) = 0.271a$, and this transcendental equation can be solved for d by iteration. The process $d_i/a = 0.5442 + (\sin(2\pi d_{i-1}/a))/2\pi$ can be applied, but it converges very slowly. In this case we have $d \simeq 0.522a$, which is in error by 4.4 percent. This simple example should however demonstrate the applicability of the method.

VII. Conclusion

The concept of nonstandard eigenvalue has been introduced and a unified variational principle in abstract operator form given, applicable for a large variety of problems. Several simple but nontrivial examples have been studied with the aid of the suggested method. It is seen that these problems, not earlier attacked with variational methods because of the nonstandard form, can be solved applying very elementary test functions and a programmable calculator for engineering accuracy.

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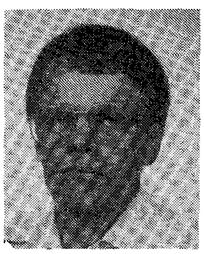

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